

## SOLUTION EXERCISE SHEET 16

**Exercise 1.** The aim of this exercise is to show the following equality

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

*Remark 0.1.* There are multiple ways how to do this. In the context of complex analysis we can for example use that  $\pi \cot(\pi z)$  has as set of poles  $\mathbb{Z}$  and that each pole is simple (this can be seen by exercise 2 in sheet 14), we can also use that  $\sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} = \frac{\pi^2}{\sin^2(\pi z)}$  or the product formula for the sinus. We are going to follow the last strategy and adapt the proof seen in class for  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .

We know by product formula of the sinus, i.e. theorem 5.28 in the lecture notes, that

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

where the product converges locally uniformly on  $\mathbb{C}$ . Now, we simply observe that  $f_N(z) := \pi z \prod_{n=1}^N \left(1 - \frac{z^2}{n^2}\right)$ , is a sequence of polynomials, which converge locally uniformly to  $\sin(\pi z)$ . Writing them as  $f_N(z) := \sum_{n=0}^{2N} a_n(f_N) z^n$ , we have by Cauchy's formula and local uniform convergence on  $\mathbb{C}$  that

$$a_5(f_N) = \frac{1}{2\pi i} \int_{\partial D_1(0)} \frac{f_N(\zeta)}{\zeta^6} d\zeta \xrightarrow{N \rightarrow \infty} \frac{1}{2\pi i} \int_{\partial D_1(0)} \frac{\sin(\pi \zeta)}{\zeta^6} d\zeta.$$

Using Cauchy's formula and the definition of sinus we know that

$$\frac{1}{2\pi i} \int_{\partial D_1(0)} \frac{\sin(\pi \zeta)}{\zeta^6} d\zeta = \frac{\pi^5}{5!} = \frac{\pi^5}{120}.$$

Further, by direct induction we can show that

$$a_5(f_N) = \pi \sum_{\substack{1 \leq k \leq N \\ k < j \leq N}} \frac{1}{k^2} \frac{1}{j^2}.$$

This being clearly a sum of non negative terms, which is bounded, we know that it converges. We get the equality

$$\pi \sum_{\substack{1 \leq k < \infty \\ k < j < \infty}} \frac{1}{k^2} \frac{1}{j^2} = \frac{\pi}{2} \left( \sum_{1 \leq k, j < \infty} \frac{1}{k^2} \frac{1}{j^2} - \sum_{n=1}^{\infty} \frac{1}{n^4} \right) = \frac{\pi}{2} \left( \frac{\pi^4}{36} - \sum_{n=1}^{\infty} \frac{1}{n^4} \right).$$

Thus we get the equality

$$\frac{\pi^4}{36} - \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{60} \iff \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{36} - \frac{\pi^4}{60} = \frac{\pi^4}{90}.$$

**Exercise 2.** Suppose first that  $f$  has a pole in  $z_*$  and let  $N \in \mathbb{N}^*$  be the order of the pole. Then clearly, we have that  $f(z) = \frac{h(z)}{(z-z_*)^N}$  on  $D_r(z_*)$  for some  $r > 0$  such that  $D_r(z_*) \subseteq U$  and some map  $h : D_r(z_*) \rightarrow \mathbb{C}$ , which is holomorphic and non zero in  $z_*$ . Thus as  $N \geq 1$  we clearly have that

$$|f(z)| = \frac{|h(z)|}{|z-z_*|^N} \xrightarrow{z \rightarrow z_*} \infty.$$

Now, for the opposite direction, suppose that  $|f(z)| \xrightarrow{z \rightarrow z_*} \infty$ . Then, we have that there exists  $r > 0$  such that  $D_r(z_*) \subseteq U$  and  $|f| > 0$  on  $D_r(z_*)^*$ . Thus on  $D_r(z_*)^*$  the map  $h := \frac{1}{f}$  is well defined and holomorphic. Further, as  $|f(z)| \xrightarrow{z \rightarrow z_*} \infty$  it is clearly bounded. Thus  $h$  has a removable singularity and can be extended holomorphically to  $D_r(z_*)$  and the extension vanishes in  $z_*$  as  $|f(z)| \xrightarrow{z \rightarrow z_*} \infty$ . Therefore, has a zero of some order  $N \in \mathbb{N}$  in  $z_*$ . As on  $D_r(z_*)^*$  we have  $f = \frac{1}{h}$  we conclude that  $f$  has a pole of  $N$ -th order in  $z_*$ .

**Exercise 3.** In order to compute the residue of the function  $\cos(z^{-1})\sin(z)$  we are going to first show a more general formula. We have the following claim

**Claim 0.2.** Let  $(a_n)_{n \in \mathbb{Z}}, (b_m)_{m \in \mathbb{Z}}$  be two functions on  $\mathbb{Z}$  taking complex values, which are both absolutely convergent, i.e.  $\sum_{n \in \mathbb{Z}} |a_n|, \sum_{m \in \mathbb{Z}} |b_m| < \infty$ . Then we have that the product series defined by  $c_k := \sum_{\substack{n, m \in \mathbb{Z} \\ k=n+m}} a_n b_m$  is absolutely convergent and we have the identity

$$\left( \sum_{n \in \mathbb{Z}} a_n \right) \cdot \left( \sum_{m \in \mathbb{Z}} b_m \right) = \sum_{k \in \mathbb{Z}} c_k.$$

*Proof.* First we observe that as

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \sum_{\substack{n, m \in \mathbb{Z} \\ k=n+m}} |a_n b_m| &= \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |a_n b_{k-n}| = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |a_n b_{k-n}| \\ &= \sum_{n \in \mathbb{Z}} |a_n| \sum_{k \in \mathbb{Z}} |b_{k-n}| = \sum_{n \in \mathbb{Z}} |a_n| \sum_{k \in \mathbb{Z}} |b_k| < \infty, \end{aligned}$$

where we used exercise 4 of exercise sheet 1 to switch the sums.

This shows in particular that  $\sum_{\substack{n, m \in \mathbb{Z} \\ k=n+m}} |a_n b_m| < \infty$ , giving that the sequence  $(c_k)_{k \in \mathbb{Z}}$  is well defined and absolutely convergent. Then using again exercise 4 of sheet 1 we get that

$$\left( \sum_{n \in \mathbb{Z}} a_n \right) \cdot \left( \sum_{m \in \mathbb{Z}} b_m \right) = \sum_{k \in \mathbb{Z}} c_k.$$

□

Next, we apply this to Laurent series to prove the product formula for Laurent series. Let again  $(a_n)_{n \in \mathbb{Z}}, (b_m)_{m \in \mathbb{Z}}$  be two functions on  $\mathbb{Z}$  taking complex values and suppose that the two Laurent series  $\sum_{n=-\infty}^{\infty} a_n z^n$  and  $\sum_{n=-\infty}^{\infty} b_n z^n$  converge on the annulus  $A(0, r, R)$  with  $0 \leq r < R \leq \infty$ . then we have that the product Laurent series defined by  $c_k := \sum_{\substack{n, m \in \mathbb{Z} \\ k=n+m}} a_n b_m$ , converges on the annulus  $A(0, r, R)$  and we have the identity

$$\left( \sum_{n=-\infty}^{\infty} a_n z^n \right) \cdot \left( \sum_{n=-\infty}^{\infty} b_n z^n \right) = \left( \sum_{n=-\infty}^{\infty} c_n z^n \right).$$

Finally, using this result, we know that the residue of  $f(z) := \cos(z^{-1}) \sin(z)$  is given by  $c_{-1} = \sum_{\substack{n, m \in \mathbb{Z} \\ -1=n+m}} a_n b_m$ , where  $(a_n)$  are the coefficients of the Laurent series of  $\cos(z^{-1})$  and  $(b_n)$  the coefficients of  $\sin(z)$ . As  $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$  and  $\cos(z^{-1}) = \sum_{n=-\infty}^0 \frac{(-1)^n}{|2n|!} z^{2n}$  we get  $\text{Res}(f, 0) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} \frac{1}{2n!}$ .

**Exercise 4.** This exercise is a direct application of the residue theorem. As we have a simple path on the circle centered at the origin and of radius 2 and  $\frac{\sin(z)}{z^2+1}$  has only two simple singularities corresponding to the zeros of  $z^2 + 1$ , we have that

$$\frac{1}{2\pi i} \int_{\partial D(0,2)} \frac{\sin(z)}{z^2+1} dz = \text{Res}\left(\frac{\sin(z)}{z^2+1}, i\right) + \text{Res}\left(\frac{\sin(z)}{z^2+1}, -i\right).$$

Making the decomposition  $\frac{\sin(z)}{z^2+1} = \frac{\sin(z)}{(z-i)(z+i)}$  we directly see that  $\text{Res}\left(\frac{\sin(z)}{z^2+1}, i\right) = \frac{\sin(i)}{2i}$  and  $\text{Res}\left(\frac{\sin(z)}{z^2+1}, -i\right) = \frac{\sin(-i)}{-2i}$ . Thus we have that

$$\frac{1}{2\pi i} \int_{\partial D(0,2)} \frac{\sin(z)}{z^2+1} dz = \frac{\sin(i)}{2i} + \frac{\sin(-i)}{-2i} = \frac{\sin(i)}{i} = \frac{e - e^{-1}}{2}.$$